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LETTER TO THE EDITOR

Homogeneous polynomial integrals of motion

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Abstract. It is shown that if a classical sytem has an integral motion which is a homogeneous polynomial in the velocities then it is also a polynomial in the position variables. The case of two degrees of freedom is investigated more closely and it is shown that the only systems which have such integrals are those with familiar momentum integrals.

In a recent paper (Xanthopoulos 1984) it was claimed that there was an intimate connection between analytic functions and constaints of motion which are homogeneous polynomials in the velocities for conservative systems with two degrees of freedom. This brief investigation will reveal, amongst other things, that unfortunately there is no such connection. In fact, quite generally for systems with m degrees of freedom such constants of motion correspond to Killing tensors of the Euclidean metric and hence are polynomial in coordinates. Moreover, at least for two degrees of freedom, the only systems of standard classical type which have integrals which are homogeneous polynomials in the velocities are those with familiar momentum integrals.

Consider a standard Hamiltonian of classical mechanics

$$H=\tfrac{1}{2}p_jp_j+V(x_i)$$

where (x_i, p_j) is a coordinate system. Suppose that f is a constant of motion for the system determined by H and that

$$f=A_{a_1\ldots a_n}p_{a_1}\ldots p_{a_n}+A_{a_1\ldots a_{n-1}}p_{a_1}\ldots p_{a_{n-1}}+\ldots A_{a_1}p_{a_1}+A$$

where $A_{a_1\ldots a_n}, A_{a_1\ldots a_{n-1}},\ldots, A_{a_1}, A$ symmetric tensors of rank $n, n-1,\ldots,1,0$ respectively. The conditions that H and f commute are easily seen to be

$$\begin{aligned}
&A_{(a_1\ldots a_n,a_{n+1})}=0\\
&A_{(a_1\ldots a_{n-1},a_n)}=0\\
&A_{(a_1\ldots a_{n-2},a_{n-1})}=nV_{,i}A_{a_1\ldots a_{n-1}i}\\
&\ldots\\
&A_{,a_1}=2V_{,i}A_{a_1i}\\
&0=V_{,i}A_i
\end{aligned}\tag{1}$$

where brackets denote symmetrisation of indices. The first two conditions in (1) define $A_{a_1\ldots a_n}$ and $A_{a_1\ldots a_{n-1}}$ as Killing tensors (Woodhouse 1975, Kalnins and Miller 1980). These were originally considered in the context of general relativity—indeed in connection with the existence of homogeneous quadratic integrals and also the separability

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of the Hamilton–Jacobi equation. In flat space they are naturally much more manageable and one has the following proposition.

Proposition. If $A_{b_1\ldots b_n}$ is a Killing tensor in a Euclidean space then $A_{b_1\ldots b_n,a_1\ldots a_n,} \equiv 0$ i.e. the Killing tensors of rank n are polynomials of degree less than or equal to n.

Proof. Suppose that for $0 \leqslant k < n$

$$A_{b_1\ldots b_k(a_1\ldots a_{n-k},a_{n-k+1}\ldots a_n)} = 0.$$

Then

$$\begin{aligned}0 &= A_{b_1\ldots b_k((a_1\ldots a_{n-k},a_{n-k+1}\ldots a_n i)b_{k+1})}\\ &= [(n-k)/(n+2)]A_{b_1\ldots b_{k+1}(a_1\ldots a_{n-k+1},a_{n-k}\ldots a_n i)}\\ &\quad +[(k+2)/(n+2)]A_{b_1\ldots b_k(a_1\ldots a_{n-k},a_{n-k+1}\ldots a_n i)b_{k+1}}.\end{aligned}$$

From the induction hypothesis, it follows that the second term on the right is zero whence so is the first. The result now follows by induction which begins successfully because $A_{b_1\ldots b_n}$ is a Killing tensor. This shows that the existence of integrals which are homogeneous polynomials in the velocities is a good deal more restrictive than Xanthopoulos (1984) has claimed. For, in that case (1) states that the polynomial is a Killing tensor, say $A_{a_1\ldots a_n}$, related also to the potential V by the conditions

$$A_{a_1\ldots a_{n-1}i}V_{,i} = 0. \tag{2}$$

What has been said so far applied equally to systems with any number of degrees of freedom. I now specialise to the case of two degrees of freedom and use coordinates x and y. In this case, the Killing tensors consist precisely of symmetrised products of the three Killing vector fields $\partial/\partial x$, $\partial/\partial y$, $y\partial/\partial x - x\partial/\partial y$. Suppose now that one asks for all systems with two degrees of freedom which possess integrals which are homogeneous of degree one. Then it is not hard to show that these integrals are necessarily of the momentum type in the sense that there exists a *canonical* coordinate system (x, y, p_x, p_y) so that the potential is either given by $V = V(x-y)$, or $V = V(x^2+y^2)$. Likewise, if one asks for systems which have homogeneous quadratic integrals independent of the Hamiltonian, one finds again that V has one of the two forms given above and that the integrals are merely the squares of momentum integrals. The latter results can be obtained directly or by specialising, for example, Thompson (1984).

I shall now investigate systems which have integrals which are homogeneous cubic polynomials. By what has been remarked earlier, this cubic integral may be assumed to be of the form

$$\begin{aligned}&A(yp_x - xp_y)^3 + 3(yp_x - xp_y)^2(B_1p_x - B_2p_y) + 3(yp_x - xp_y)(C_1p_x^2 + 2C_3p_xp_y + C_2p_y^2)\\ &\quad + D_1p_x^3 - 3D_3p_x^2p_y^2 + 3D_4p_xp_y^2 - D_2p_y^3\end{aligned}$$

where A, B_1, B_2, C_1, C_2, C_3, D_1, D_2, D_3, D_4 are constants. This integral could be written in classical tensor notation as $A_{ijk}p_ip_jp_k$. (1) now gives three conditions relating the A_{ijk} and V. In order that these be compatible to yield non-constant V it is necessary that

$$A_{111}A_{122} - A_{112}^2 = 0 \tag{3}$$

$$A_{222}A_{112} - A_{122}^2 = 0 \tag{4}$$

$$A_{111}A_{222} - A_{112}A_{122} = 0. \tag{5}$$

When these conditions are imposed one finds that once more that they can be satisfied essentially only in the two ways mentioned above i.e. after a suitable change of coordinates

$$V=V(x-y), \qquad V=V(x^2+y^2). \tag{6}$$

Indeed, this is almost obvious from just (5). The preceding argument may be generalised to the case of arbitrary n but this shall not be done because the calculations become rather cumbersome. Thus, one is led to conclude that the only standard classical systems with two degrees of freedom which have integrals which are homogeneous polynomials in the momenta are of the type give by (6)—a result which obviously conflicts with those given by Xanthopoulos (1984).

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